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INVOLUTORY QUARTIC TRANSFORMATIONS IN SPACE OF FOUR DIMENSIONS

BY

NINA ALDERTON

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INVOLUTORY QUARTIC TRANSFORMATIONS IN SPACE OF FOUR DIMENSIONS

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§1. Cayley in his paper "On the Rational Transformation between two Spaces"[1] gives a general discussion of the quadric transformation between two planes and the cubo-cubic transformation between two spaces. The cubic transformation in space was further studied by F. R. Morris[2] who gives an analytic treatment of the cases in which the Jacobian, the sextic curve whose points go into lines by means of this transformation, breaks up into curves of lower degree, one or more of which are straight lines. A synthetic treatment of the general case of the cubic space transformation is given by D. N. Lehmer[3] in his paper "On Combinations of Involutions," and of six of the special cases by Elizabeth J. Easton.[4]

§2. A discussion of the general involutory quartic transformation in space of four dimensions has been given by P. H. Schoute[5] in an article, "La Surface de Jacobi d'un systeme lineaire d'hyperquadriques $Q^3{}_2$ dans l'espace E^4 a quatre dimensions." The writer was not acquainted with this article when work upon the present paper was begun and consequently began with a general consideration of the involutory transformation by means of four hyperquadrics. Schoute makes his transformation with respect to a pencil of hyperquadrics and defines the transform P' of the point P as being the intersection of the polar spaces of P with respect to the triple infinitude of hyperquadrics of the pencil. This is equivalent, however, to using four hyperquadrics, for four independent hyperquadrics determine the pencil. The present paper gives the discussion of the general involutory quartic transformation with respect to four hyperquadrics, as originally planned, before going on to a consideration of the cases in which the Jacobian, now a surface, breaks up into

[1] Proc. London Math. Soc., vol. 3 (1869-1873), pp. 127-180.

[2] "Classification of Involutory Cubic Space Transformations," Univ. Calif. Publ. Math., vol. 1, pp. 223-240.

[3] Am. Math. Monthly, vol. 18, no. 3 (March 1911). Also Steiner's Ges. Werke, vol. 2, p. 651.

[4] Ms., Master's Thesis, "Certain Special Cubo-cubic Space Transformations," 1917, in Univ. Calif. Library, Dept. of Mathematics.

[5] Archives du Musée Teyler, série 2, 7, 1900-01.

surfaces of lower degree including at least one plane. Although it has seemed advisable to examine the subject analytically as well as synthetically to a considerable extent, the synthetic treatment only will be presented in most cases in this paper.

§3. NOTATION

The exponent of the symbol of a surface will be used to denote the infinitude of points on the surface and the subscript to denote the degree of the surface; thus S_2^3 designates a quadric hypersurface.

§4. DEFINITIONS

1. All of the 3-spaces through a line form what we shall call an axial pencil of 3-spaces. All of the 3-spaces through a plane form a plane pencil of 3-spaces. There are ∞^2 3-spaces in an axial pencil and ∞ in a plane pencil.

2. Harmonic 3-spaces of a plane pencil are four 3-spaces which are cut by any line in four harmonic points.

3. The polar 3-space of a point P with respect to a hyperquadric is the locus of a point which is the fourth harmonic to P and the two points in which any line through P cuts the hyperquadric.

4. The simplex of reference in 4-space is a figure bounded by five 3-spaces. The five 3-spaces intersect two at a time in ten planes, three at a time in ten lines, and four at a time in five points which are the vertices of the simplex.

§5. PRELIMINARY THEOREMS

1. In a plane pencil (§4, 1) of 3-spaces, if four planes are cut by one line in four harmonic points they are cut by every line in four harmonic points.

Proof.—Upon any line p take four harmonic points. These points together with the plane of the plane-pencil determine four harmonic 3-spaces; for, cut across the plane-pencil by any 3-space through p. We then have an axial pencil of planes and we know that planes corresponding to four harmonic points of the line are four harmonic planes which are cut by any line in four harmonic points. Since this is true in any 3-space through p, we see that any line cuts the four 3-spaces corresponding to four harmonic points of the line in four harmonic points. Hence as a point P moves along p, the polar 3-spaces of P with respect to four hyperquadrics will form four projective plane-pencils. Similarly, as P moves over a plane, the polar 3-spaces of P will form four projective axial pencils.

2. There are ∞^2 lines cutting four planes in 4-space. Proof: Call the planes a_1, a_2, a_3, a_4. If we pass a 3-space through a_1, it will cut a_2, a_3, and a_4 in three lines. There will be ∞ lines in the 3-space cutting these three lines, and these lines will also cut a_1, since they lie in the same 3-space with a_1. Hence in every 3-space through a_1, there will be ∞ lines cutting a_1, a_2, a_3 and a_4. But there are ∞ 3-spaces about a_1 (§4, 1) and consequently ∞^2 lines cutting a_1, a_2, a_3 and a_4.

Case I

GENERAL TRANSFORMATION BY MEANS OF FOUR HYPERQUADRICS

§6. We may set up an involutory one-to-one correspondence between the points of 4-space by means of four arbitrarily chosen hyperquadrics. To a point P corresponds a point P', the intersection of the four polar 3-spaces of P with respect to the four hyperquadrics. Since the polar 3-spaces of P' must all pass through P and since four 3-spaces can intersect, in general, in only one point, the point P also corresponds to the point P' and we have an involutory, one-to-one correspondence.

§7. Thus, in general, to a point P will correspond a point P', but there are certain points to which correspond a whole line of points. The locus of such a point is the Jacobian. We shall show that

Theorem I. The locus of all points whose transform is a line is a surface of the tenth degree in 4-space, $J^2{}_{10}$. (2, §3). Proof: Let the four hyperquadrics be

$$\begin{aligned} A &= \Sigma a_{ij} x_i x_j = 0 \\ B &= \Sigma b_{ij} x_i x_j = 0 \\ C &= \Sigma c_{ij} x_i x_j = 0 \\ D &= \Sigma d_{ij} x_i x_j = 0 \end{aligned} \qquad \begin{bmatrix} i = 1 \ldots 5 \\ j = 1 \ldots 5 \end{bmatrix}$$

The polar 3-spaces of a point P with respect to A, B, C and D are then

$$\begin{aligned} x'_1 A_1 + x'_2 A_2 + x'_3 A_3 + x'_4 A_4 + x'_5 A_5 &= 0 \\ x'_1 B_1 + x'_2 B_2 + x'_3 B_3 + x'_4 B_4 + x'_5 B_5 &= 0 \\ x'_1 C_1 + x'_2 C_2 + x'_3 C_3 + x'_4 C_4 + x'_5 C_5 &= 0 \\ x'_1 D_1 + x'_2 D_2 + x'_3 D_3 + x'_4 D_4 + x'_5 D_5 &= 0 \end{aligned}$$

Ordinarily these four 3-spaces will intersect in a point. If, however, the equations are linearly dependent they will intersect in a line. The condition for this is that the matrix of the coefficients be of rank three; i.e. that all of the four-rowed determinants of the matrix vanish. Hence from the matrix

$$\left\| \begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ B_1 & B_2 & B_3 & B_4 & B_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 \\ D_1 & D_2 & D_3 & D_4 & D_5 \end{matrix} \right\|$$

we shall have five four-rowed determinants equal to zero. Each of these equations represents a quartic hypersurface. The Jacobian is the locus of points lying on all five of these hyperquartics. The hyperquartic we get by omitting the fourth column and the one we get by omitting the fifth column will intersect in a surface of degree sixteen, $S^2{}_{16}$. But they have in common the matrix formed of the first three columns which represents a surface of the sixth degree through which the other hyperquartics do not pass. Hence the Jacobian, the surface through which all five hyperquartics pass, is a $J^2{}_{10}$.

§8. *Theorem II.* The lines which are the transforms of the points of $J^2{}_{10}$ form a ruled hypersurface, $j^3{}_{15}$.

Proof: If we denote by $X=0$, $Y=0$, $Z=0$, $W=0$, and $V=0$ the hyperquartics of the matrix of §7 which we get by omitting the first column, then the second, etc., we know that the equation of the Jacobian hypersurface, which is made up of the lines which are the transforms of points of $J^2{}_{10}$, may be found by equating to zero the determinant of the partial derivatives of X, Y, Z, W and V with respect to x_1, x_2, x_3, x_4 and x_5; thus

$$j=\begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ Y_1 & Y_2 & Y_3 & Y_4 & Y_5 \\ Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \\ W_1 & W_2 & W_3 & W_4 & W_5 \\ V_1 & V_2 & V_3 & V_4 & V_5 \end{vmatrix} = 0$$

Each element of this determinant is of degree three in x_1, x_2, x_3, x_4 and x_5 and hence the equation represents a hypersurface of degree fifteen which we shall designate as $j^3{}_{15}$.

§9. Thus we see that the points of 4-space go by this transformation into other points with the exception of points on a $J^2{}_{10}$ whose points transform into the lines of a ruled $j^3{}_{15}$.

§10. It seems at first thought as though there might be points in 4-space whose polar 3-spaces meet in planes, but this is not true in general. The condition for this would be that the matrix of the coefficients of the four polar 3-spaces of §7 be of rank two. The forty cubic hypersurfaces obtained by setting each three-rowed determinant equal to zero would all have to pass through the points which transform into planes. Taking the first three rows of the matrix,

$$\left|\begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ B_1 & B_2 & B_3 & B_4 & B_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 \end{matrix}\right|$$

the three cubic hypersurfaces represented by columns 1, 2, 3; 1, 2, 4; and 1, 2, 5 intersect in a cubic surface and a curve. The other seven cubic hypersurfaces represented by this matrix will not pass through the cubic surface but will pass through the curve. Similarly, taking the matrix

$$\left|\begin{matrix} B_1 & B_2 & B_3 & B_4 & B_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 \\ D_1 & D_2 & D_3 & D_4 & D_5 \end{matrix}\right|$$

we find that the ten cubic hypersurfaces whose equations are the different three-rowed determinants of the matrix set equal to zero pass through another curve. Two curves do not, in general, intersect in 4-space and therefore the twenty cubic hypersurfaces so far examined have no points in common and hence the forty have none. Consequently there are no points whose transforms are planes when the hyperquadrics are unrelated.

§11. *Theorem III.* If a point P moves along a line p, its corresponding point P' moves along a quartic curve in 4-space.

Proof: As the point P moves along the line p, its polar 3-spaces with respect to the four hyperquadrics revolves about four planes forming what we shall call plane pencils of 3-spaces. These four plane pencils are projective pencils for there is a one-to-one correspondence between the points of p and the 3-spaces of the four plane pencils, and to four harmonic points of the line correspond four harmonic 3-spaces of the pencils. (2, §4, and Theorem I, §5.) The locus of the intersection of corresponding 3-spaces of the four projective plane pencils will be the transform of the line p. If we cut across the plane pencils by a 3-space, we have four projective axial pencils of planes in a 3-space. Four and only four sets of corresponding planes of these pencils meet in points, (Reye, Geometrie der Lage, vol. 2, XII, p. 93), so there are four points of the locus in every 3-space and hence the transform of the line p is a quartic curve in 4-space. The single infinitude of points of the curve corresponds to the single infinitude of points of the line p.

§12. *Theorem IV.* If the point P moves over a 3-space, the corresponding point P' moves over a quartic hypersurface.

Proof: If the point P moves over a 3-space, the polar 3-spaces of P with respect to the four hyperquadrics revolve about four points forming what we shall call four points of 3-spaces. There will be a triple infinitude of points P' corresponding to the triple infinitude of points P and hence points P' lie on a hyper-surface. In order to find the degree of this hypersurface, cut across it by a line. Transforming, the line goes into a quartic curve, as we have just seen, and the hypersurface back into the 3-space, giving off also the $j^3{}_{15}$. The hypersurface must give off the $j^3{}_{15}$ when transformed, for it contains $j^2{}_{10}$, since the 3-space of which it is the transform cuts all of the lines of $j^3{}_{15}$. The points of the hypersurface which do not transform into lines but into points will go back into the points of the 3-space on account of the involutory relation between the points under this transformation. But the quartic curve cuts the 3-space in four points. Therefore the line cuts the hypersurface in four points and it is a quartic hypersurface.

§13. *Theorem V.* If P moves over a plane, its corresponding point P' moves over an $S_6{}^2$.

Proof: The plane may be considered as the intersection of two 3-spaces, R_1 and R_2. When we transform, these two 3-spaces go into two quartic hypersurfaces which intersect in a surface $S^2{}_{16}$. But we have seen (§12) that every quartic hypersurface which is the transform of a 3-space must pass through $J^2{}_{10}$. Hence $J^2{}_{10}$ is a part of $S^2{}_{16}$ and the remaining part is an $S_6{}^2$ which is then the transform of the plane.

§14. *Theorem VI.* The multiplicity of lines of $j^3{}_{15}$ through points of $J^2{}_{10}$ and the multiplicity of points of $J^2{}_{10}$ on lines of $j^3{}_{15}$ is four.

Proof: If p' is a line which is the transform of a point P of $J^2{}_{10}$, then the polar 3-spaces of all points of p' will pass through P. Ordinarily the four polar 3-spaces

of points of p' intersect in only one point and this must be the point P. Since P is always a common point of four polar 3-spaces of points of p', in order for p' to transform into a quartic curve it must go into four lines passing through P. But P is any point of $J^2{}_{10}$ and hence the Jacobian must be a surface of $j^3{}_{15}$ of multiplicity four. The points of p' all go into the point P except four points whose transforms are the four lines through P. Hence there are four points of $J^2{}_{10}$ on every line of $j^3{}_{15}$, or the lines of $j^3{}_{15}$ cut $J^2{}_{10}$ in four points.

§15. Summary.—The principal facts which have been established concerning the general involutory quartic transformation in 4-space are: that lines go into quartic curves, planes into surfaces of the sixth degree, and 3-spaces into quartic hypersurfaces; also, that the locus of points whose transforms are lines is a surface of the tenth degree, and the lines which are the transforms of these points form a hypersurface of the fifteenth degree.

Case II

ONE FUNDAMENTAL HYPERQUADRIC IS A SPACE-PAIR

§16. (a) In Case I the four hyperquadrics of the transformation were perfectly general. We shall now consider the case in which one of the hyperquadrics A is a space-pair whose 3-spaces intersect in a plane a_1. It is evident that we may still set up a one-to-one correspondence between the points of 4-space, for ordinarily to a point P will correspond a point P', the intersection of polar 3-spaces of P with respect to B, C and D and the 3-space conjugate with respect to the 3-spaces of A to that determined by P and the plane a_1.

§17. The polar 3-space with respect to A of a point P on a_1 is indeterminate and hence to such a point corresponds the line of intersection of the polar 3-spaces with respect to B, C and D. Hence a_1 is a part of $J^2{}_{10}$ and

Theorem VII. The Jacobian is a plane and an $S^2{}_9$ when one of the fundamental hyperquadrics is a space-pair.

§18. *Theorem VIII.* The Jacobian hypersurface is an $S^3{}_3$ and an $S^3{}_{12}$ when one of the fundamental hyperquadrics is a space-pair.

Proof: The transforms of points of a_1 are lines of $j^3{}_{15}$. As P moves over a_1, the polar 3-spaces of P revolve about three lines forming three projective axial pencils. (1, §5). Now three projective axial pencils of 3-spaces intersect in a hypersurface of the third degree, for if we cut across them by any 3-space we have three points of planes intersecting in a cubic surface. Hence $j^3{}_{15}$ breaks down into a hypersurface of the third degree and one of the twelfth degree.

§19. (b) If the two 3-spaces of A coincide; i.e. if A is composed of two coincident 3-spaces we cannot set up an involutory relation between points of 4-space, for the polar 3-space of any point will be A itself and the points of 4-space will transform into those of a 3-space.

Case III

TWO FUNDAMENTAL HYPERQUADRICS ARE SPACE-PAIRS

§20. (a) Suppose A and B are the two space-pairs and a_1 and a_2, the planes of intersection of the pairs of 3-spaces of A and B respectively, have only a point in common. It is evident that we may still set up an involutory one-to-one correspondence between the points of 4-space. The planes a_1 and a_2 are part of $J^2{}_{10}$ and we have the theorem

Theorem IX. The Jacobian is composed of two planes and an $S_8{}^2$ when two of the fundamental hyperquadrics are space-pairs.

§21. The planes a_1 and a_2 both go into cubic hypersurfaces of $j^3{}_{15}$. Hence

Theorem X. The Jacobian hypersurface is composed of two $S^3{}_3$'s and an $S^3{}_9$ when two of the fundamental hyperquadrics are space-pairs.

§22. There is a plane lying on both of the cubic hypersurfaces and hence forming part of their intersection; namely the transform of the point of intersection of the two planes a_1 and a_2. This point transforms into a plane since its polar 3-spaces with respect to A and B are indeterminate, and its transform is then the intersection of its polar 3-spaces with respect to C and D.

§23. A line l cutting either a_1 or a_2 will transform into a line and a cubic curve. Suppose it cuts a_1. The line l_1 is the transform of the point where the given line l cuts a_1. To get the transform of the rest of l, allow the point P to move along l. The point P and the plane a_1 always determine the same 3-space and hence all points of l have the same polar 3-space with respect to A. The polar 3-spaces of points of l with respect to B, C and D form the three projective plane pencils of 3-spaces and these intersect in lines of a ruled $S^2{}_3$, for if we cut across them by a 3-space we have three projective axial pencils whose corresponding planes intersect in points of a cubic curve. The polar 3-space of points of l with respect to A will cut this $S_3{}^2$ in a twisted cubic. Hence l transforms into a line and a twisted cubic lying in the polar 3-space with respect to A of points of l. Similarly, a line cutting a_2 will transform into a line and a twisted cubic lying in the polar 3-space with respect to B of points of the line.

§24. A line l cutting both a_1 and a_2 will transform into two lines l_1 and l_2 and a conic C_2 lying in the plane of intersection of the two polar 3-spaces of points of l with respect to A and B.

§25. Other lines which do not transform into quartic curves are those lying on a_1 (or a_2). A line lying on a_1 (or a_2) and not passing through P_1, the intersection of a_1 and a_2, becomes an $S_3{}^2$, the locus of intersections of corresponding 3-spaces of three projective plane pencils of 3-spaces. If the line passes through this point P_1, also, the cubic surface breaks up into a plane, the transform of P_1, and an $S_2{}^2$ lying in the polar 3-space with respect to B (or A) of points of the line.

§26. The surface S_6^2 which is the transform of a plane has two lines lying on it, the transforms of the points of intersection of the plane with a_1 and a_2. If these two points coincide at P_1, the S_6^2 breaks down into a plane which is the transform of P_1 and an $S^2{}_5$. Further degeneration of the S_6^2 occurs when the plane intersects a_1 (or a_2) and both a_1 and a_2 in lines.

§27. A 3-space containing a_1 (or a_2) will transform into a ruled cubic hypersurface which is the transform of a_1 (or a_2) and the 3-space which is the polar of points of the 3-space with respect to A (or B).

(b) Now suppose A and B are so related that a_1 and a_2 intersect in a line L_3; i.e. that all of the 3-spaces of A and B pass through L_3. Then we have the theorem

Theorem XI. The Jacobian is composed of three planes and a $S^2{}_7$ when space-pairs A and B have a line in common.

Proof: The points of L_3 transform into the planes of a planed hyperquadric which are the intersections of two plane pencils of polar 3-spaces of points of L_3 with respect to C and D. Hence the plane a_1 (or a_2) will transform into the planes of a planed hyperquadric and ∞^2 lines of the 3-space which is the polar of points of a_1 (or a_2) with respect to B (or A). Hence a_1 and a_2 are still parts of $J^2{}_{10}$ and have a line lying on them whose points transform into planes. Now a_1 and a_2 determine a 3-space since they have a line in common. Points of this 3-space will have a single polar 3-space with respect to A and a single polar 3-space with respect to B and these two 3-spaces will intersect in a plane β_3 passing through L_3. The polar 3-space of points of β_3 with respect to both A and B will be the 3-space determined by a_1 and a_2, and this 3-space will cut the polar 3-spaces with respect to C and D in lines. Hence β_3 is also a part of $J^2{}_{10}$, as are the planes a_1 and a_2.

§28. *Theorem XII.* The Jacobian hypersurface is composed of a planed hyperquadric counted twice, three 3-spaces, and an S_8^3 when the four 3-spaces of A and B have a line in common.

Proof: The plane a_1 (or a_2) goes into a hyperquadric which is the transform of L_3 and the 3-space which is the polar 3-space with respect to B (or A) of points of a_1 (or a_2). The plane β_3 goes into the 3-space determined by a_1 and a_2. Hence three 3-spaces and a hyperquadric counted twice will be part of the $j^3{}_{15}$. It should be noted that a line on $J^2{}_{10}$ whose transform is counted twice is a triple line on $J^2{}_{10}$; in this case the three planes a_1, a_2, and β_3 pass through L_3.

§29. Again, lines which have a special position with respect to a_1 and a_2 will not transform as usual into quartic curves. Any line cutting L_3 will transform into a plane and a conic lying in the plane of intersection of the two polar 3-spaces of points of the line with respect to A and B. The transform of a plane cutting L_3 or passing through L_3 will be a degenerate S_6^2. The 3-space determined by a_1 and a_2 will go into the planed hyperquadric and the two 3-spaces which are the remainder of the transforms of a_1 and a_2. All points of the 3-space not on a_1 or a_2 go into points of β_3.

§30. (c) Suppose one of the 3-spaces of B (or A) passes through the plane of intersection of the 3-spaces of A (or B). If A and B are so related that one of the

3-spaces, R_3 of B, passes through a_1, then a_1 and a_2 determine R_3 and must intersect in a line L_3 as in (b). The plane β_3 will now coincide with a_1, the intersection of the conjugate 3-space of R_3 with respect to A and R_3 itself, since it is self-conjugate with respect to B. Hence

Theorem XIII. The Jacobian is composed of a single plane, a double plane, and an S_7^2 when one of the 3-spaces of B (or A) passes through a_1 (or a_2).

§31. *Theorem XIV.* The Jacobian hypersurface is composed of a planed hyperquadric counted twice, a single 3-space, a double 3-space and an S_8^3. The single 3-space is the conjugate with respect to A of R_3 and the double one is R_3 itself.

§32. (d) If a_1 and a_2 coincide in a_1, the points of 4-space go into points of a_1 and we no longer have a one-to-one correspondence.

Case IV

THREE FUNDAMENTAL HYPERQUADRICS ARE SPACE-PAIRS

§33. (a) Suppose A, B and C are space-pairs and their planes, a_1, a_2, and a_3 respectively, have only points in common.

Theorem XV. The Jacobian is composed of three planes and an S_7^2 when three of the fundamental hyperquadrics are space-pairs. The planes a_1, a_2 and a_3 which form part of the $J^2{}_{10}$ intersect two at a time in three points P_1, P_2 and P_3. These points transform into planes. There are two such points on each of the three planes.

§34. *Theorem XVI.* The Jacobian hypersurface is composed of three hypercubics and an S_6^3 when three of the fundamental hyperquadrics are space-pairs.

The three hypercubics are transforms of the planes a_1, a_2 and a_3.

§35. If a_1 and a_2 intersect in P_3, a_2 and a_3 in P_1, and a_3 and a_1 in P_2, then a line such as the one joining P_1 with any point of a_1 transforms into a plane and two lines. A plane through two of the points P_1, P_2, P_3, say P_1 and P_2, will go into two planes which are the transforms of P_1 and P_2, another plane which is the transform of the other points of the line $P_1 P_2$, and a cubic surface lying in the 3-space conjugate to that determined by the plane to be transformed and a_3. The plane P_1, P_2, P_3, goes into six planes.

§36. (b) Suppose two of the three planes a_1, a_2 and a_3, say a_1 and a_2, intersect in a line L_3.

Theorem XVII. The Jacobian is composed of four planes and an $S^2{}_6$ when two of the three fundamental space-pairs A, B and C have a line in common.

The four planes are the planes a_1, a_2 and a_3 and the plane β_3 which is the intersection of the two polar 3-spaces of points of the 3-space determined by a_1 and a_2 with respect to A and B.

§37. *Theorem XVIII.* The Jacobian hypersurface is composed of three 3-spaces, a planed hyperquadric counted twice, a hypercubic and an S_5^3 when two of the three fundamental space-pairs A, B, C have a line in common.

§38. (c) Suppose the three fundamental space-pairs A, B, C are so related that a 3-space R_3 of B passes through α_1. This implies that α_1 and α_2 have a line L_3 in common.

Theorem XIX. The Jacobian is composed of two planes, a double plane, and an S_6^2 when three of the fundamental hyperquadrics are space-pairs and a 3-space of one of them passes through the plane of another.

This is true since two of the four planes of Theorem XVII now coincide in α_1. (Compare Theorem XIII.)

§39. *Theorem XX.* The Jacobian hypersurface is composed of one single 3-space, one double 3-space, a planed hyperquadric counted twice, a hypercubic and an S_5^3 when three of the fundamental hyperquadrics are space-pairs and a 3-space of one of them passes through a plane of another.

§40. (d) The three fundamental space-pairs may be so related that A and B have a line L_3 in common and B and C have a line L_1 in common. In this case

Theorem XXI. The Jacobian is composed of six planes and an S_4^2 when A and B have a line in common and B and C have a line in common.

The six planes are α_1, α_2, α_3, the plane β_3 which is the intersection of the two polar 3-spaces of points of the 3-space R' determined by α_1 and α_2 with respect to A and B, the plane β_1 which is the intersection of the two polar 3-spaces of points of the 3-space R'' determined by α_2 and α_3 with respect to B and C, and the plane β_2 which is the intersection of the polar 3-space of points of R' with respect to A and the polar 3-space of points of R'' with respect to C.

§41. The lines L_3 and L_1 intersect since they both lie in α_2 and this point P which lies on each of the three planes α_1, α_2 and α_3 transforms into a 3-space. It should be noted that the planes β_1, β_2 and β_3 also pass through P and hence a point on $J^2{}_{10}$ whose transform is a 3-space of $j^3{}_{15}$ counted three times has six sheets of the surface passing through it. The 3-space into which all points of α_1 except those lying on L_3 transform is the 3-space determined by α_2 and α_3, and similarly for α_3. This may be shown by taking A, B and C as space-pairs through three planes of the simplex of reference. (4, §4.) The 3-spaces determined by α_1 and α_2 and by α_2 and α_3 are then two of the five faces of the simplex. Hence in this case

Theorem XXII. The Jacobian hypersurface is composed of one triple 3-space, four double 3-spaces and an S_4^3 when the three fundamental space-pairs A, B and C are so related that A and B have a line in common and B and C have a line in common. Two of the double 3-spaces are of course the 3-spaces determined by α_1 and α_2 and by α_2 and α_3.

§42. Any line lying in α_2 and not passing through P transforms into three planes since it cuts both L_1 and L_3. If it passes through P it transforms into a 3-space and a plane.

§43. (e) Let one of the 3-spaces R_3 of B pass through a_1 while a_2 and a_3 still have a line L_1 in common.

Theorem XXIII. The Jacobian is composed of four single planes, a double plane and an S_4^2 when one of the 3-spaces of B passes through a_1 and B and C have a line in common.

This is due to the coincidence of two of the planes of Theorem XXI, the double plane being the plane a_1.

§44. *Theorem XXIV.* The Jacobian hypersurface is composed of two triple 3-spaces, two double 3-spaces, a single 3-space and an S_4^3 when one of the 3-spaces of B passes through a_1 and B and C have a line in common.

The 3-space R_3 is one of the triple 3-spaces and the plane determined by a_2 and a_3 is the single 3-space.

§45. (f) Suppose one of the 3-spaces R_3 of B passes through a_1 and one of the 3-spaces R_5 of C passes through a_2. Then

Theorem XXV. The Jacobian is composed of two single planes, two double planes, and an S_4^2 when one of the 3-spaces of B passes through a_1 and one of the 3-spaces of C passes through a_2. The two double planes are a_1 and a_2 while a_3 is one of the single planes.

§46. *Theorem XXVI.* The Jacobian hypersurface is composed of a triple 3-space, four double 3-spaces, and an S_4^3 when one of the 3-spaces of B passes through a_1, and one of the 3-spaces of C passes through a_2.

The 3-spaces R_3 and R_5 are double 3-spaces.

§47. (g) Suppose one of the 3-spaces, R_3, of B passes through a_1 and the other, R_4, passes through a_3.

Theorem XXVII. The Jacobian is composed of two single planes, two double planes and an S_4^2 when one 3-space of B passes through a_1 and the other through a_3.

The planes a_1 and a_3 are double planes while a_2 is now a single plane.

§48. *Theorem XXVIII.* The Jacobian hypersurface is composed of a triple 3-space, four double 3-spaces, and an S_4^3 when one 3-space of B passes through a_1 and the other through a_3. R_3 and R_4 are double 3-spaces.

§49. (h) Let the planes a_1, a_2, and a_3 intersect in lines two at a time.

Theorem XXIX. The Jacobian is composed of six planes and an S_4^2 when the planes a_1, a_2 and a_3 intersect two at a time in lines.

Call the intersection of a_1 and a_2 L_3, the intersection of a_2 and a_3 L_1, and the intersection of a_3 and a_1 L_2. The planes a_1, a_2 and a_3 now lie in a 3-space R and intersect in a point P so the lines L_1, L_2 and L_3 are concurrent. The 3-space R will have conjugate 3-spaces with respect to A, B and C which will intersect two at a time in planes β_3, β_1 and β_2 which together with planes a_1, a_2 and a_3 are the six planes of the $J^2{}_{10}$.

§50. *Theorem XXX.* The Jacobian hypersurface is composed of one triple 3-space, four double 3-spaces and an S_4^3 when the planes α_1, α_2 and α_3 intersect two at a time in lines. This is true since the plane α_1 transforms into lines of β_1 and the three 3-spaces which are the transforms of L_2 and L_3, α_2 into lines of β_2 and the three 3-spaces which are the transforms of L_1 and L_3, and α_3 into lines of β_3 and the three 3-spaces which are the transforms of L_1 and L_2.

§51. The transform of points of R with respect to A, B and C will be three 3-spaces intersecting in a line L_4. Hence the planes β_1, β_2 and β_3 all transform into R, the transform of points of L_4, which is then one of the double 3-spaces of $J^3{}_{15}$. (§28.) Points of this line L_4 will transform into planes of R. Hence there are four lines, L_1, L_2, L_3 and L_4, whose transforms are 3-spaces of planes.

§52. Any line in α_1, α_2 or α_3 will transform into three planes if it does not pass through P, for it will cut two of the lines L_1, L_2 and L_3. If it does pass through P it will transform into a 3-space and a plane through one of the lines L_1, L_2 or L_3. The planes β_1, β_2 and β_3 all pass through L_4 and hence any line in β_1, β_2 or β_3 will transform into two planes since it cuts L_4 and either L_1, L_2 or L_3.

§53. (i) Suppose one of the 3-spaces R_3 of B in (h) passes through α_1. The planes α_1, α_2 and α_3 now all lie in R_3 and hence R_3 passes through α_3. Hence part of $J^2{}_{10}$ is α_1 taken twice, α_3 taken twice, α_2 and β_2 or

Theorem XXXI. The Jacobian is composed of two double planes, two single planes, and an S_4^2 when a 3-space R_3 of B passes through α_1 and α_3.

§54. *Theorem XXXII.* The Jacobian hypersurface is composed of one triple 3-space, four double 3-spaces, and an S_4^3 when a 3-space R_3 of B passes through α_1 and α_2.

The 3-space R_3 is one of the double 3-spaces.

§55. (j) If L_2 coincides with L_1, then L_3 also coincides with L_1. In this case all of the points of 4-space go into points of L_1 and we no longer have a one-to-one correspondence.

Case V

THE FOUR FUNDAMENTAL HYPERQUADRICS ARE SPACE-PAIRS

§56. (a) Let A, B, C and D be four space-pairs any two of which have only a point in common. The planes α_1, α_2, α_3, and α_4 of A, B, C and D respectively are planes of the $J^2{}_{10}$. Hence

Theorem XXXIII. The Jacobian is composed of four planes and an S_6^2 when the four fundamental hyperquadrics are space-pairs intersecting in pairs in points.

§57. *Theorem XXXIV.* The Jacobian hypersurface is composed of four hypercubics and an S_3^3 when the four fundamental hyperquadrics are space-pairs intersecting in pairs in points.

§58. There are ∞^2 lines cutting a_1, a_2, a_3 and a_4. (2, §5.) Any one of these lines will transform into four lines which are rulings of the $j^3{}_{15}$. It cuts $J^2{}_{10}$ four times so the point into which the remainder of the line transforms is the point through which the four lines pass. Hence the four lines are concurrent. Lines cutting only three of the planes a_1, a_2, a_3 and a_4 also transform into four lines but these lines are not concurrent.

§59. (b) Suppose the 3-spaces of A and B have a line L_3 in common.

Theorem XXXV. The Jacobian is composed of five planes and an $S_5{}^2$ when two of the four fundamental space-pairs have a line in common.

The planes are a_1, a_2, a_3 and a_4 and the plane β_3 which transforms into lines of the 3-space determined by a_1 and a_2.

§60. *Theorem XXXVI.* The Jacobian hypersurface is composed of three 3-spaces, a planed hyperquadric counted twice, two cubic hypersurfaces and an $S_2{}^3$.

§61. (c) Suppose a 3-space R_3 of B passes through a_1.

Theorem XXXVII. The Jacobian is composed of three single planes, a double plane, and an $S_5{}^2$ when a 3-space of B passes through a_1.

The double plane is the plane a_1.

§62. *Theorem XXXVIII.* The Jacobian hypersurface is composed of a single 3-space, a double 3-space, a planed hyperquadric counted twice, and an $S_2{}^3$, when a 3-space of B passes through a_1. The double 3-space is R_3.

§63. (d) Suppose A and B have a line L_3 in common and C and D a line L_2 in common.

Theorem XXXIX. The Jacobian is composed of six planes and an $S_4{}^2$ when A and B have a line in common and C and D have a line in common.

The planes are a_1, a_2, a_3, a_4 and also β_3 and β_2 which transform into the 3-spaces determined by a_1 and a_2 and by a_3 and a_4 respectively.

§64. *Theorem XL.* The Jacobian hypersurface is composed of two hyperquadrics counted twice and seven 3-spaces when A and B have a line in common and C and D have a line in common.

§65. (e) If the lines L_2 and L_3 of (d) intersect in a point P, then every point in 4-space transforms into the point P and we no longer have a one-to-one correspondence.

§66. (f) Suppose the space-pairs A, B, and C are so related that A and B have a line L_3 in common and B and C have a line L_1 in common.

Theorem XLI. The Jacobian is composed of seven planes and an $S_4{}^2$ when A and B have a line in common and B and C have a line in common.

The planes are a_1, a_2, a_3 and a_4 and the planes β_1, β_2 and β_3 of Theorem XXI.

§67. *Theorem XLII.* The Jacobian hypersurface is composed of a triple 3-space, four double 3-spaces, a single 3-space, and a hypercubic.

§68. (g) Suppose the space-pairs A and B have a line L_3 in common, B and C a line L_1 in common, and C and D a line L_2 in common.

Theorem XLIII. The Jacobian breaks down into the ten planes of the simplex of reference (4, §4) when the four fundamental hyperquadrics are space-pairs and three pairs of them intersect in lines. This is more easily seen analytically. Take as the fundamental space-pairs

$$A = x^2_1 - bx^2_2 = 0$$
$$B = x^2_1 - cx^2_3 = 0$$
$$C = x^2_3 - dx^2_4 = 0$$
$$D = x^2_4 - ex^2_5 = 0$$

The Jacobian turns out to be, in addition to the planes α_1, α_2, α_3 and α_4, the planes

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \quad \begin{cases} x_1 = 0 \\ x_4 = 0 \end{cases} \quad \begin{cases} x_3 = 0 \\ x_5 = 0 \end{cases} \quad \begin{cases} x_1 = 0 \\ x_5 = 0 \end{cases} \quad \begin{cases} x_2 = 0 \\ x_4 = 0 \end{cases} \quad \text{and} \quad \begin{cases} x_2 = 0 \\ x_4 = 0 \end{cases}$$

§69. *Theorem XLIV.* The Jacobian hypersurface breaks down into the five faces of the simplex of reference counted three times when the four fundamental hyperquadrics are space-pairs and the three pairs of them intersect in lines.

The equation of the Jacobian hypersurface turns out to be

$$x^3_1 \cdot x^3_2 \cdot x^3_3 \cdot x^3_4 \cdot x^3_5 = 0$$

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